ACCURATE UNCERTAINTY PROPAGATION THROUGH NONLINEAR SYSTEMS. P. Singla[1], T. Singh[1], P. Scott[2], G. Terejanu[2], Y. Cheng[1] and K. V. Umamaheswara Reddy[1], [1]Department of Mechanical & Aerospace Engineering, University at Buffalo, Buffalo, NY-14260, [2]Department of Computer Science and Engineering, University at Buffalo, Buffalo, NY-14260.

Introduction: Mathematical models are approximate representations of physical processes and consequently have uncertainties associated with them. Furthermore, no sensor is perfect. Sensor measurements are generally some linear/nonlinear combination of states and are usually corrupted with quantization errors, superimposed noise, etc. Propagating the states of a process using an uncertain model initialized with erroneous initial conditions will result in forecasts with growing uncertainties. In practice, if this uncertainty is not ameliorated, the useful spatio-temporal prediction domain is quite limited. Hence, it is a logical step to assimilate sensor measurements with model prediction to reduce the associated uncertainties of the desired estimates.

This approach had its birth with the development of the Kalman Filter for linear systems [1]. Subsequently various researchers have endeavored to exploit knowledge of statistics, dynamic systems theory and numerical analysis to develop techniques which apply to the various classes of problems of interest. For instance, the Extended Kalman and the Unscented Filter [2] were developed for low order nonlinear systems, while the Ensemble Kalman [3] and the Reduced Rank Square Root filters [4] were developed for mildly nonlinear large scale systems such as those studied in meteorology, hydrology etc. For low order nonlinear systems, several approximate methods [5-9] most popular being the particle filter, Monte Carlo methods, Gaussian closure (or higher order moment closure), Equivalent Linearization, Polynomial Chaos, and Stochastic Averaging have been gaining increasing attention. However, all these approaches provide only an approximate charaterization of the uncertainty propagation and are only suitable for low or moderate dimension problems.

The exact description of this problem for continuous systems is provided by the well known Fokker Planck Kolmogorov Equation (FPKE), the solution to which contains complete information about the state PDF. While the FPKE holds the key for uncertainty propagation through nonlinear dynamical systems, it is a formidable problem to solve because of the issues like positivity, normality, discretization, and most importantly, the dimensionality of the system. For low dimensioned systems (eg. < 4), the FPKE can be solved numerically after discretization, but for high-dimensioned systems (eg. > 3) advanced numerical methods are required due to the exponential growth of computational work with increasing number of dimensions (Bellman's well-known "curse of dimensionality"). In particular, there is need for efficient, robust and accurate methods for solving the FPKE for exact propagation of uncertainties through complex nonlinear systems and assessing their effect on system performance.

The main contribution in our approach is solving the FPKE efficently to study the uncertainty propagation through complex nonlinear systems. Analytical and closed-form solutions to the FPE exist only for a very small class of simplified low-dimensional dynamic systems. Various approximate methods such as statistical linearization , Gaussian closure, residual projection methods, and finite element (FEM) based methods have been used for the approximate solution of FPKE. A good discussion of these methods can be found in Refs. [10-12], in which the finite difference and FEM numerical techniques have been used to solve the FPE for structure mechanics problems. Although the conventional FEM is generally assumed to be well suited for solving PDEs, the generation of meshes – especially in spaces with dimensions greater than three, still remains a stumbling block in its success. Furthermore, there are no convenient means of incorporating prior knowledge about the solution being sought. Consequently, in the present context, while solving the FPE with the conventional FEM, one cannot make use of the fact that most PDEs have exponential functional form. As a consequence, most of these numerical methods suffer from the curse of dimensionality and there is a need for adaptive methods that can solve FPE in higher dimensions (> 3) in an efficient fashion.

In our recent work, we introduced novel methods for accurate uncertainty propagation through a general nonlinear system. The basic idea of this approach is to approximate the model parameter pdf and state pdf by a weighted average of sufficient number of distinct local Gaussian approximations in the *stochastic space.* In conventional methods, the weights of the components of a Gaussian mixture are kept constant while propagating the uncertainty through a nonlinear system and are updated only in the presence of measurement data [7]. This strategy is valid if the underlying dynamics is linear or the system is at worst marginally nonlinear. This is not the case for general nonlinear systems, and updated estimates of weights are required for accurate propagation of the state pdf. However, the existing literature provides no means for adaptation of the weights of different Gaussian components in the mixture model during the propagation of state pdf. The lack of adaptive algorithms for weights of Gaussian mixture is a serious disadvantage of existing algorithms and provide the motivation for the proposed work.

Here we investigate two schemes to update the weights corresponding to different components of Gaussian mixture models for accurate uncertainty propagation. The first updates the weights by constraining the Gaussian sum approximation to satisfy the Fokker-Planck equation for continuous-time dynamical systems. The second method updates the forecast weights by minimizing the integral square difference between the true forecast pdf and its Gaussian sum approximation. Both methods lead to a convex quadratic programming problem which has been shown to be computationally efficient [14,15]. Further, we shall use the Bayesian framework for the assimilation of sensor data with the model output. These novel ideas are validated against and compared with the existing methods such as Particle Filter, Ensemble Kalman filter etc. Fig. 1 shows the solution obtained by solving the FPKE corresponding to nonlinear oscillator given by the following equation:

$$\ddot{x} + \eta\dot{x} + x + \omega^2\left(x^2 + \dot{x}^2\right)\dot{x} = \Gamma(t)$$

Notice that although the initial distribution is Gaussian in nature, the distribution rapidly evolves into the shape of a volcano just after 15 seconds with the rim of volcano near the deterministic limit cycle. Our experience shows that this kind of behavior is difficult to capture with conventional methods, but the proposed methods yield effective state estimates.

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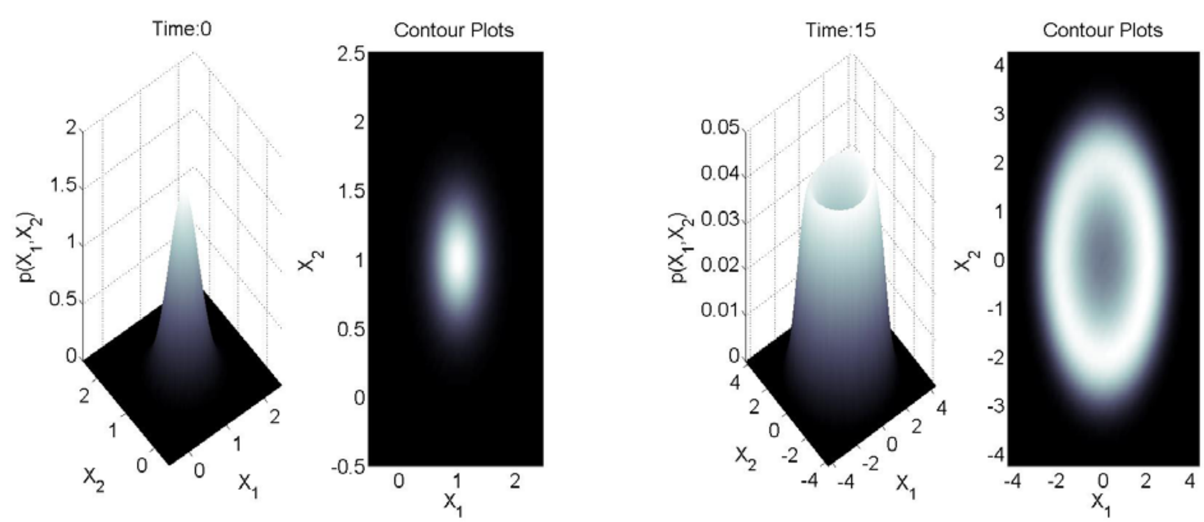

Figure 1: Uncertainty Propagation Through Nonlinear Oscillator